Gayatri Vidya Parishad College of Engineering for Women (A)

I B.Tech. II Semester Regular/ Supplementary Exams, July 2026

LINEAR ALGEBRA AND VECTOR CALCULUS

[Common to CSE, CSE-AIML, CSE-CS, ECE, EEE and IT]

KEY & Scheme of Evaluation

1. a) $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ $\begin{matrix} R_2 : R_2 - 2R_1 \\ R_3 : R_3 - 4R_1 \\ R_4 : R_4 - 4R_1 \end{matrix} \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$

$\begin{matrix} R_3 : R_3 - R_2 \\ R_4 : R_4 - 3R_2 \end{matrix} \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$\rho(A)$ = No. of non zero rows $= 2$

1 b) $[A|B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$ $\begin{matrix} R_2 : 2R_2 - 7R_1 \\ R_3 : R_3 - R_1 \end{matrix} \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix}$

(i) No solution if $\lambda = 5$ & $\mu \neq 9$ [$\because \rho(A) = 2$ & $\rho(A|B) = 3$]

(ii) Unique Solution if $\lambda \neq 5$ [$\because \rho(A) = \rho(A|B) = 3$]

(iii) Infinite No. of solutions if $\lambda = 5$ & $\mu = 9$ [$\because \rho(A) = \rho(A|B) = 2 < 3$]

2. a) $[A|B] = \begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$ $R_1 \leftrightarrow R_3$ $\begin{bmatrix} 1 & 4 & 9 & 16 \\ 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \end{bmatrix}$

$\begin{matrix} R_2 : R_2 - 3R_1 \\ R_3 : R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 4 & 9 & 16 \\ 0 & -10 & -24 & -30 \\ 0 & -7 & -17 & -22 \end{bmatrix}$ $R_3 : 10R_3 - 7R_2$ $\begin{bmatrix} 1 & 4 & 9 & 16 \\ 0 & -10 & -24 & -30 \\ 0 & 0 & -2 & -10 \end{bmatrix}$

$x + 4y + 9z = 16$

$-10y - 24z = -30$

$-2z = -10$

By back substitution, $z = 5,\ y = -9,\ x = 7$

$\therefore X = \begin{bmatrix} 7 \\ -9 \\ 5 \end{bmatrix}$

2b) Let $A = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = LU$

$\Rightarrow \quad A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$

Consider $AX = B \Rightarrow LUX = B$

Let $UX = Y$, Then $LY = B$

Solving $LY = B$ for Y, we get $Y = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$

Solving $UX = Y$ for X, we get $X = \begin{bmatrix} 7/8 \\ 9/8 \\ -1/8 \end{bmatrix}$

3 a) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Characteristic eqn. of A is $|A - \lambda I| = 0 \Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$

Eigen values of A are $-2, 3, 6$

Corresponding eigen vectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

3 b) $A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{bmatrix}$

Characteristic eqn. of A is $\lambda^3 - 11\lambda^2 + 122 = 0$

Cayley Hamilton theorem statement

$A^2 = \begin{bmatrix} 16 & 20 & 34 \\ 2 & 33 & 50 \\ -2 & 28 & 72 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 54 & 220 & 374 \\ 22 & 241 & 550 \\ -22 & 308 & 670 \end{bmatrix}$

Proving $A^3 - 11A^2 + 122I = 0$

$A^{-1} = \frac{1}{122}(11A - A^2)$

$= \frac{1}{122} \begin{bmatrix} 17 & 24 & -23 \\ 20 & -22 & 16 \\ -9 & 16 & 5 \end{bmatrix}$

4. a) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Char. eqn. of A is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Eigen values of A are $\lambda = 1, 2, 3$.

Corresponding eigen vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \& \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Statement : If λ is an eigenvalue of A with X as the eigen vector then $\frac{1}{\lambda}$ is an eigen value of A^{-1} with the same eigen vector X "

$\therefore$ The eigen values of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$ and the Corresponding eigen vectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \& \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

4 b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Char. eqn. of $A^T A$ is $|A^T A - \lambda I| = 0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\Rightarrow \lambda = 3, 1$

Singular values of A are $\sigma_1 = \sqrt{3}, \sigma_2 = 1$

$$A_{3\times 2} = U_{3\times 3} \Sigma_{3\times 2} V^T_{2\times 2}$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Eigen vector Corr. to $\lambda = 3$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\therefore v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

" " " $\lambda = 1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $\therefore v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Now $U = [u_1\ u_2\ u_3]$ where $u_i = \frac{1}{\sigma_i} A v_i$

$$u_1 = \frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\1\end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Supposing $u_3 = \begin{bmatrix}a\\b\\c\end{bmatrix}$ to be orthogonal with u_1 & u_2 we get

$$u_3 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$$

Final expression $A = U\Sigma V^T = \ldots$

5) $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_1x_3$

Matrix of the QF is $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

Char. eqn. of A is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 8\lambda + 2 = 0$$

There are no integer roots for this equation exists.

6) $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Characteristic eqn. of A is $\lambda^3 - \lambda^2 - 5\lambda + 5 = 0$

$\Rightarrow \lambda = 1, +\sqrt{5}, -\sqrt{5}$

Eigen vectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} \sqrt{5}-1 \\ 1 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} -\sqrt{5}-1 \\ -1 \\ 1 \end{bmatrix}$ resp.

$P = \begin{bmatrix} -1 & \sqrt{5}-1 & -\sqrt{5}-1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ & $P^{-1} = \frac{-1}{2\sqrt{5}} \begin{bmatrix} 0 & 2\sqrt{5} & 2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -2+\sqrt{5} & -1 \end{bmatrix}$

$P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$

$A^4 = PD^4P^{-1} = \begin{bmatrix} 25 & 24 & 24 \\ 0 & 25 & 0 \\ 0 & -24 & 1 \end{bmatrix}$

7. a) $\bar{f} = yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k}$

$\text{Curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \bar{i}\,(x-x) - \bar{j}\,(y-y) + \bar{k}\,(z-z) = \bar{0}$

$\therefore \bar{f}$ is irrotational.

Consider $\bar{f} = \nabla\phi \Rightarrow yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$\Rightarrow \frac{\partial\phi}{\partial x} = yz, \quad \frac{\partial\phi}{\partial y} = zx, \quad \frac{\partial\phi}{\partial z} = xy$

We know that $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$\Rightarrow d\phi = yz\,dx + zx\,dy + xy\,dz$

$\Rightarrow \phi = 3xyz + C$

7 b) $f = x^2 - y^2 + 2z^2$, $\nabla f = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$

At $(1,2,3)$, $\nabla f = 2\bar{i} - 4\bar{j} + 12\bar{k}$

$\overline{PQ} = \overline{OQ} - \overline{OP} = 4\bar{i} - 2\bar{j} + \bar{k} = \bar{a}$ (say)

Directional derivative $= \nabla f \cdot \frac{\bar{a}}{|\bar{a}|} = \frac{28}{\sqrt{21}}$

8 a) $\phi = x^2 + y^2 + z^2 - 29$ $\qquad \psi = x^2 + y^2 + z^2 + 4x - 6y - 8z - 47$

$\nabla \phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$ $\qquad \nabla \psi = (2x+4)\bar{i} + (2y-6)\bar{j} + (2z-8)\bar{k}$

At the point $(4,-3,2)$, $\nabla \phi = 8\bar{i} - 6\bar{j} + 4\bar{k} = \bar{a}$

$\nabla \psi = 12\bar{i} - 12\bar{j} - 4\bar{k} = \bar{b}$

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|.|\bar{b}|} = \frac{152}{\sqrt{116}\sqrt{304}} = \frac{152}{\sqrt{35264}} = \frac{152}{8\sqrt{551}} = \frac{19}{\sqrt{551}}$$

8 b) $$\nabla^2 r^n = \sum \frac{\partial^2}{\partial x^2} r^n$$

$$= \sum \frac{\partial}{\partial x}\left[n r^{n-1} \frac{\partial r}{\partial x}\right]$$

$$= \sum n \frac{\partial}{\partial x}(x r^{n-2}) \qquad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= \sum n \left[1.r^{n-2} + x(n-2) r^{n-3}\frac{\partial r}{\partial x}\right]$$

$$= \sum n r^{n-2} + n(n-2)x^2 r^{n-4}$$

$$= 3n r^{n-2} + n(n-2) r^2 r^{n-4}$$

$$= (3n + n^2 - 2n) r^{n-2}$$

$$= n(n+1) r^{n-2}$$

9. a) $$\bar{F}\cdot d\bar{r} = (2y+3)\,dx + zx\,dy + (yz-x)\,dz$$
$$= (8t^2+12t+2t^5+3t^6-6t^4)\,dt$$

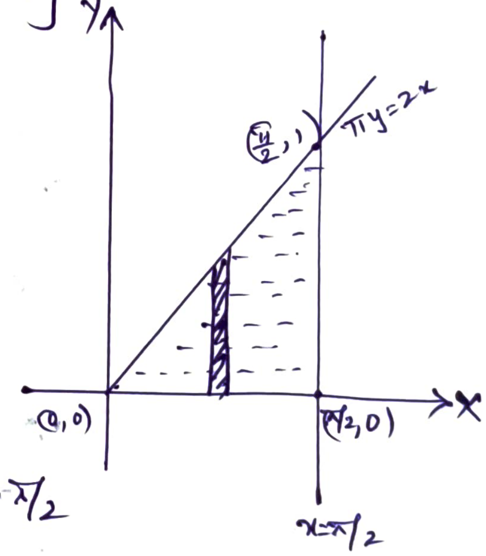
$$\text{Work Done} = \int_A^B \bar{F}\cdot d\bar{r}$$
$$= \int_{t=0}^{1} (8t^2+12t+2t^5+3t^6-6t^4)\,dt$$
$$= \frac{8}{3}+6+\frac{1}{3}+\frac{3}{7}-\frac{6}{5} = \frac{864}{105} = \frac{288}{35}$$

b) Green's theorem: $$\int_C M\,dx+N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$M = y-\sin x$, $\quad N = \cos x$

$\dfrac{\partial M}{\partial y} = 1$, $\quad \dfrac{\partial N}{\partial x} = -\sin x$. $\quad \therefore \dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y} = -(1+\sin x)$

$$\int_C M\,dx+N\,dy = -\iint_R (1+\sin x)\,dx\,dy$$
$$= -\int_{x=0}^{\pi/2}\int_{y=0}^{2x/\pi} (1+\sin x)\,dy\,dx$$
$$= -\frac{2}{\pi}\int_{x=0}^{\pi/2} (x+x\sin x)\,dx$$
$$= -\frac{2}{\pi}\left[\frac{x^2}{2}-x\cos x+\sin x\right]_0^{\pi/2}$$
$$= -\frac{2}{\pi}\left[\frac{\pi^2}{8}-\frac{\pi}{2}\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\right]$$
$$= -\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$$

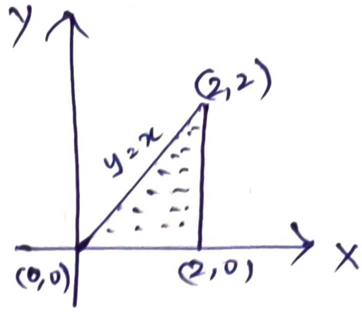


10. a) Stokes theorem: $$\int_C \bar{F}\cdot d\bar{r} = \int_S \text{Curl}\,\bar{F}\cdot\bar{n}\,dS$$

$$\bar{F} = 2y^2\,\bar{i} + 3x^2\,\bar{j} - (2x+z)\,\bar{k}$$

$$\text{Curl}\,\bar{F} = 2\,\bar{j} + (6x-4y)\,\bar{k}$$

As the given triangle is completely in XY-plane, $\bar{n} = \bar{k}$



$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_S (6x-4y)\,dS$$

$$= \int_{x=0}^{2}\int_{y=0}^{x} (6x-4y)\,dy\,dx$$

$$= \int_{x=0}^{2} 4x^2\,dx$$

$$= 32/3$$

10. b) Divergence Theorem states that $\int_S \bar{F}\cdot\bar{n}\,dS = \int_V \operatorname{div}\bar{F}\,dV$

$$\operatorname{div}\bar{F} = 4-4y-2z$$

$$\therefore \int_S \bar{F}\cdot\bar{n}\,dS = \int_V (4-4y-2z)\,dV$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y-2z)\,dz\,dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-12y)\,dy\,dx$$

$$= 3\times 4\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} dy\,dx \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 0 \text{ if } f \text{ is odd} \atop = 2\int_0^a f\,dx \text{ if even}\right]$$

$$= 12\int_{x=0}^{2}\sqrt{4-x^2}\,dx$$

$$= 12\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 12\left[2(\sin^{-1}1 - \sin^{-1}0)\right]$$

$$= 12\pi$$

Prepared by: B. Bharathi

Dr. B. Bharathi

Assistant Professor,

Dept. of BS&H,

GVPCEW (A).